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NUMERICAL EXPERIMENTS ON TRANSVERSE VIBRATIONS OF A RECTANGULAR PLATE OF GENERALIZED ANISOTROPY WITH A FREE EDGE

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1. INTRODUCTION

Filamentary composite materials are commonly used in several fields of modern technology.

This fact has prompted the need of a sound understanding of the static and dynamic behaviour of anisotropic structural elements.

In the case of plates or slabs of generalized anisotropy executing small amplitude, transverse vibrations one must make use of approximate analytical or numerical methods in a great majority of situations. Excellent analytical and experimental studies have been performed in this problem area [1–5].

In view of the difficulty in satisfying natural boundary conditions, e.g. null bending moments normal to hinged edges or zero Kirchhoff force in the case of a free edge,† it seems that the Rayleigh–Ritz method constitutes one of the most appropriate techniques for dealing with transverse vibrations of anisotropic plates. As it will be shown in the next section, another inherent difficulty when dealing with the energy functional corresponding to vibrating anisotropic plates is the fact that "popular" coordinate functions, when solving isotropic and orthotropic plate problems, may yield null contributions when performing the required integrations over the rectangular domain and depending upon the boundary conditions.

The present paper deals with some numerical experiments when determining the fundamental frequency of transverse vibration of the anisotropic plates shown in Figure 1 when edge 2 is free and for the following combinations of boundary conditions for the remaining edges: (a) edges 1, 3 and 4 simply supported; (b) edges 1, 3 and 4 clamped; and (c) edges 1 and 4 simply supported, edge 3 clamped.

2. THE GOVERNING ENERGY FUNCTIONAL AND APPROPRIATE COORDINATE FUNCTIONS

Using Lekhnitskii's standard notation [7] and in the case of normal modes the governing functional is expressed as

$$J[W] = \frac{1}{2}\iint\left[D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_{\sigma\sigma}\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2 + 4D_{1\sigma}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial x\,\partial y} + 4D_{2\sigma}\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x\,\partial y}\right]\mathrm{d}x\,\mathrm{d}y - \tfrac{1}{2}\rho h\omega^2\iint W^2\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

† This condition is, in general, seldom satisfied when using beam functions or polynomial coordinate functions in the case of isotropic and orthotropic plates [6].

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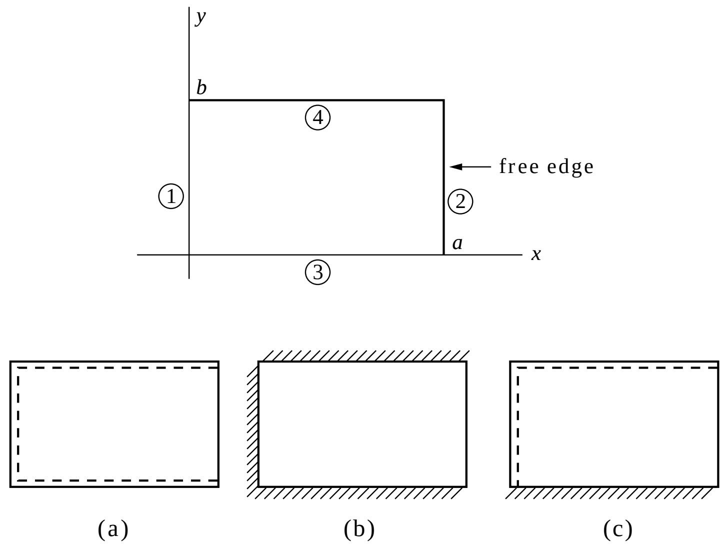

Figure 1. Anisotropic plates executing transverse vibrations considered in the present study. (a) Case (a), (b) Case (b) and (c) Case (c).

where $W(x, y)$ is the plate displacement amplitude and ω is the circular frequency corresponding to one of the normal modes of the plate. The present study is concerned with the determination of the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h / D_{22}}\omega_1 a^2$ of the rectangular plates shown in Figure 1.

If one approximates $W(x, y)$ by an expression of the type

$$W(x, y) \simeq W_I(x, y) = \left[\sum_{j=1}^{N} A_j X_j(x)\right] Y(y) \tag{2}$$

one observes that for cases (a) (b) and (c) (Figure 1) and since $Y(o) = Y(b) = 0$, the term containing $D_{1\sigma}$ in expression (1) vanishes once the integration

$$D_{1\sigma} \iint \frac{\partial^2 W_I}{\partial x^2} \frac{\partial^2 W_I}{\partial x \, \partial y} \, \mathrm{d}x \, \mathrm{d}y = 0 \tag{3}$$

is performed.

In an analogous fashion, when considering case (b), one has

$$\frac{\partial W_I}{\partial y}(x, 0) = \frac{\partial W_I}{\partial y}(x, b) = 0. \tag{4}$$

Consequently,

$$\frac{\mathrm{d}Y(o)}{\mathrm{d}y} = \frac{\mathrm{d}Y(b)}{\mathrm{d}y} = 0, \tag{5}$$

and this implies that

$$\iint \frac{\partial^2 W_I}{\partial y^2} \frac{\partial^2 W_I}{\partial x \, \partial y} \, \mathrm{d}x \, \mathrm{d}y = 0. \tag{6}$$

Accordingly the contribution of $D_{2\sigma}$ disappears for this arrangement of boundary conditions.

It becomes quite clear then that a functional relation of type (2) will not describe appropriately the physical phenomenon under study and that it will be convenient to use a function $W_{II}(x, y)$ where at least some of the intervening terms do not lead to situations (3) or (6). Some examples will be shown in the next section.

3. NUMERICAL EXAMPLES

Determinations of Ω_1 have been performed for cases (a), (b) and (c) of Figure 1 for the following anisotropic mechanical characteristics [1]: $D_{11}/D_{22} = 0{\cdot}21396$, $D_{12}/D_{22} = 0{\cdot}3249$, $D_{16}/D_{22} = 0{\cdot}1690$, $D_{2\sigma}/D_{22} = 0{\cdot}5117$, $D_{\sigma\sigma}/D_{22} = 0{\cdot}3387$.

Case (*a*). Following reference [8] one employs

$$W_I(x, y) = \left(\sum_{j=1}^{N} \sin\frac{\pi x}{\gamma_j a}\right)\sin\frac{\pi y}{b} \tag{7a}$$

with $\gamma_1 > 1$,

$$W_{II}(x, y) = \left(\sum_{j=1}^{N} \sin\frac{\pi x}{\gamma_j a} + (x - y)\sin\frac{\pi x}{\gamma_N a}\right)\sin\frac{\pi y}{b} \tag{7b}$$

with $\gamma_1 > 1$ and $N > 1$ and where the γs are Rayleigh's optimization parameters.

Case (*b*)

$$W_1(x, y) = \left(\sum_{j=1}^{N} A_j \sin^2\frac{\pi x}{\gamma_j a}\right)(y^2 - 2y^3 + y^4) \tag{8a}$$

with $\gamma_1 > 1$,

$$W_{II}(x, y) = \left(\sum_{j=1}^{N-1} A_j \sin^2\frac{\pi x}{\gamma_j a} + A_N(x - y)\sin^2\frac{\pi x}{\gamma_N a}\right)(y^2 - 2y^3 + y^4) \tag{8b}$$

with $\gamma_1 > 1$ and $N > 1$.

Case (*c*)

$$W_I(x, y) = \left(\sum_{j=1}^{N} A_j \sin\frac{\pi x}{a\gamma_j}\right)(y^2 - \tfrac{5}{3}y^2 + \tfrac{2}{3}y^4) \tag{9a}$$

with $\gamma_1 > 1$,

$$W_{II}(x, y) = \sum_{j=1}^{N-1} A_j \sin\frac{\pi x}{\gamma_j a} + A_N(x - y)\sin\frac{\pi x}{\gamma_N a}\Bigg)(y^2 - \tfrac{5}{3}y^3 + \tfrac{2}{3}y^4) \tag{9b}$$

with $\gamma_1 > 1$ and $N > 1$.

Substituting expressions (7), (8) or (9) in the governing functional and requiring that

$$\frac{\partial J}{\partial A_j}[W] = 0 \tag{10}$$

TABLE 1

Values of Ω_1 for the anisotropic configuration shown in Figure 1(a)

	$a/b = 5/2$	3/2	1	2/3	2/5
Isotropic plate [6]	63·287	24·009	11·685	6·0937	3·008
W_I(1 term)*	63·676	24·150	11·725	6·087	2·984
W_I(2 terms)*	63·660	24·138	11·71	6·080	2·981
W_I(3 terms)*	63·000	23·650	11·40	5·92	2·93
W_{II}(2 terms)†	62·673	23·036	10·619	5·318	2·690
W_{II}(3 terms)‡	61·662	22·497	10·496	5·020	2·448

* $W_I(x, y) = \sum A_j \sin \frac{\pi x}{\gamma_1 a} \sin \frac{\pi y}{b}$ with $\gamma_1 > 1$.

† $W_{II}(x, y) = A_1 \sin \frac{\pi x}{\gamma_1 a} \sin \frac{\pi y}{b} + A_2(x - y) \sin \frac{\pi x}{\gamma_2 a} \sin \frac{\pi y}{b}$.

‡ $W_{II}(x, y) = \left(A_1 \sin \frac{\pi x}{\gamma_1 a} + A_2 \sin \frac{\pi x}{\gamma_2 a} + A_3(x - y) \sin \frac{\pi x}{\gamma_3 a}\right) \sin \frac{\pi y}{b}$.

one obtains a homogeneous, linear system of equations in the A_js. A determinantal equation is finally obtained from the non-triviality condition, its lowest root being the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h/D_{22}}\omega_1 a^2$. Since Ω_1 is an upper bound, by minimizing it with respect to the optimization parameters γ_j, one obtains an optimized value of Ω_1.

Tables 1, 2 and 3 depict values of fundamental frequency coefficients for cases (a), (b) and (c) of Figure 1, respectively. The first line of each table contains the eigenvalues of isotropic plates, for comparison purposes.

In all cases the fundamental frequencies of the anisotropic structural elements have been computed using (1) W_I with one, two and three terms and (2) W_{II} with two and three terms. One immediately observes the fact that the use of the functional relation W_{II} influences

TABLE 2

Values of Ω_1 for the anisotropic configuration shown in Figure 1(b)

	$a/b = 5/2$	3/2	1	2/3	2/5
Isotropic plate [6]	141·106	51·783	24·020	11·880	6·024
W_I(1 term)*	141·963	52·166	24·108	11·642	5·260
W_I(2 terms)*	141·241	51·517	23·567	11·246	5·036
W_I(3 terms)*	140·889	51·356	23·529	11·243	5·022
W_{II}(2 terms)†	141·88	52·092	24·040	11·571	5·156
W_{II}(3 terms)‡	140·253	50·558	22·735	10·610	4·643

* $W_I(x, y) = \left[\sum_{j=1}^{J} A_j \sin^2 \frac{\pi x}{a\gamma_j}\right](y^2 - 2y^3 + y^4), \qquad j = 1, 2.$

† $W_{II}(x, y) = \left(A_1 \sin^2 \frac{\pi x}{\gamma_1 a} + A_2(x - y) \sin^2 \frac{\pi x}{\gamma_2 a}\right)(y^2 - 2y^3 + y^4).$

‡ $W_{II}(x, y) = \left(A_1 \sin^2 \frac{\pi x}{\gamma_1 a} + A_2 \sin^2 \frac{\pi x}{\gamma_2 a} + A_3(x - y) \sin^2 \frac{\pi x}{\gamma_3 a}\right)(y^2 - 2y^3 + y^4).$

TABLE 3

Values of Ω_1 for the anisotropic configuration shown in Figure 1(c)

	$a/b = 5/2$	3/2	1	2/3	2/5
Isotropic plate [6]	97·806	36·150	16·865	8·2400	3·6907
W_I(1 term)*	100·69	38·067	18·162	9·053	4·089
W_I(2 terms)*	100·519	37·871	17·987	8·930	4·034
W_I(3 terms)*	100·443	37·787	17·920	8·891	4·024
W_{II}(2 terms)†	99·334	36·523	17·351	8·754	4·030
W_{II}(3 terms)‡	98·287	36·265	16·868	8·214	3·661

* $W_I(x, y) = \left[\sum_{j=1}^{J} A_j \sin \frac{\pi x}{a\gamma_j}\right](y^2 - \frac{5}{3}y^3 + \frac{2}{3}y^4)$.

† $W_{II}(x, y) = A_1 \sin \frac{\pi x}{\gamma_1 a}(y^2 - \frac{5}{3}y^3 + \frac{2}{3}y^4) + A_2 \sin \frac{\pi x}{\gamma_2 a}(x - y)(y^2 - \frac{5}{3}y^3 + \frac{2}{3}y^4)$.

‡ $W_{II}(x, y) = \left(A_1 \sin \frac{\pi x}{\gamma_1 a} + A_2 \sin \frac{\pi x}{\gamma_2 a} + A_3 \sin \frac{\pi x}{\gamma_2 a}(x - y)\right)(y^2 - \frac{5}{3}y^3 + \frac{2}{3}y^4)$.

considerably the determination of the frequency coefficients, specially for the cases (a) and (b) considered in Tables 1 and 2.

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